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Sub-Riemannian geometry of the coefficients of univalent functions ☆

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Abstract

We consider coefficient bodies $\mathcal{M}_n$ for univalent functions. Based on the Löwner–Kufarev parametric representation we get a partially integrable Hamiltonian system in which the first integrals are Kirillov's operators for a representation of the Virasoro algebra. Then $\mathcal{M}_n$ are defined as sub-Riemannian manifolds. Given a Lie–Poisson bracket they form a grading of subspaces with the first subspace as a bracket-generating distribution of complex dimension two. With this sub-Riemannian structure we construct a new Hamiltonian system to calculate regular geodesics which turn to be horizontal. Lagrangian formulation is also given in the particular case $\mathcal{M}_3$.

Keywords: Univalent function; Coefficient; Hamiltonian system; Distribution of a tangent bundle; Sub-Riemannian manifold; Geodesics

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1. Introduction

Let U be the unit disk $U = \{z\colon |z| < 1\}$. Let $\mathbf{S}$ stand for the standard class of holomorphic univalent functions $f : U \to \mathbb{C}$ normalized by

$$f(z) = z\left(1 + \sum_{n=1}^{\infty} c_n z^n\right), \quad z \in U.$$

By $\tilde{\mathbf{S}}$ we denote the class of functions from $\mathbf{S}$ smooth (C^{∞}) on the boundary S^1 of U. Considering $\{c_1, \ldots, c_n, \ldots\}$ as local affine coordinates on $\mathbf{S}$ or $\tilde{\mathbf{S}}$ we provide an embedding of these infinite-dimensional manifolds into $\mathbb{C}^{\mathbb{N}}$. We denote by $\mathcal{M} \subset \mathbb{C}^{\mathbb{N}}$ the limiting set for the coefficient bodies $\mathcal{M} = \lim_{n\to\infty} \mathcal{M}_n$, where

$$\mathcal{M}_n = \bigl\{(c_1, \ldots, c_n)\colon\ f \in \tilde{\mathbf{S}}\bigr\}.$$

The class $\mathbf{S}$ is compact regarding to the local uniform topology in U and $\tilde{\mathbf{S}}(\simeq \mathcal{M})$ is a dense subclass of $\mathbf{S}$. By the famous de Branges' result [8] (former Bieberbach conjecture), $\mathcal{M}$ lies in the bounded domain $|c_n| < n + 1$, $n \geqslant 1$. The set $\mathcal{M}_1$ is the open disk $|c_1| < 2$. However, the description of $\mathcal{M}_n$ is extremely difficult for $n > 1$. Only the first non-trivial coefficient body $\mathcal{M}_2$ has been described completely by Schaeffer and Spencer in 1950 in their well-known monograph [29]. A qualitative description of $\mathcal{M}_n$, $n \geqslant 2$, has been partially given in [4]. Apart from these two monographs there are only few works where a progress in such a problem has been made (see, e.g., [26,27]). Such a complicated nature of the coefficient bodies in the Euclidean structure of $\mathbb{C}^n$ encourages us to think of other pertinent geometries suitable to the structure of $\mathcal{M}_n$.

On the other hand, the manifold $\mathcal{M}$ is a natural representation of Kirillov's infinite-dimensional Kählerian manifold $\operatorname{Diff} S^1/S^1$ through conformal welding, here $\operatorname{Diff} S^1$ denotes the Lie group of orientation preserving diffeomorphisms of the unit circle S^1, and S^1 is the subgroup of rotations associated with S^1. Indeed, given a map $f \in \tilde{\mathbf{S}}$ we construct an adjoint univalent meromorphic map

$$g(z) = d_1 z + d_0 + \frac{d_{-1}}{z} + \cdots,$$

defined in the exterior U^* of U, and such that $\hat{\mathbb{C}} \setminus \overline{f(U)} = g(U^*)$. This gives the identification $\operatorname{Diff} S^1/S^1$ with $\mathcal{M}$, see [1,17]. The central extension of $\operatorname{Diff} S^1$ by $\mathbb{R}$ is the Virasoro–Bott group. The corresponding central extension of the space $\operatorname{Vect} S^1$ of vector fields on S^1 is the Virasoro algebra ($= \operatorname{Vect} S^1 \oplus \mathbb{R}$). The infinitesimal action of $\operatorname{Diff} S^1$ on $\mathcal{M}$ (given by the Goluzin–Schiffer variation) leads to special vector fields L_j on $\mathcal{M}$, Kirillov's operators for a representation of the Virasoro algebra.

We deduce a Hamiltonian system for the Löwner–Kufarev trajectories in $\mathcal{M}_n$. In view of Hamiltonian mechanics, this formulation performs a trivial motion with constant speed (and vanishing energy). Our aim is to describe a sub-Riemannian structure of the n-complex-dimensional manifold $\mathcal{M}_n$ based on Kirillov's operators and to describe geodesics in this structure. We calculate them explicitly for $n = 3$. Such a description gives a non-trivial motion in which the energy of the system conserves along non-Riemannian geodesics.

In our setup Kirillov's operators appear as the first integrals of a partially integrable Hamiltonian system for c_n generated by the Löwner–Kufarev representation of univalent functions. The sub-Riemannian structure is based on the distribution defined by only two first vector fields L_1 and L_2 and other vector fields form a grading sequence. The horizontal curves are only of finite length in the corresponding sub-Riemannian metric and we give a description of regular geodesics in $\mathcal{M}$. Lagrangian formulation is also given in the particular case $\mathcal{M}_3$.

2. Hamiltonian system for the coefficients

2.1. Coefficient bodies

By the *coefficient problem for univalent functions* we mean the problem of precise finding the regions $\mathcal{M}_n$ defined above. These sets have been investigated by a great number of authors, but the most remarkable source is a monograph [29] written by Schaeffer and Spencer in 1950. Among other contributions to the coefficient problem we distinct a monograph by Babenko [4] that contains a good collection of qualitative results on the coefficient bodies $\mathcal{M}_n$. The results concerning the structure and properties of $\mathcal{M}_n$ include (see [4,29]):

(i) $\mathcal{M}_n$ is homeomorphic to a $(2n-2)$-dimensional ball and its boundary $\partial\mathcal{M}_n$ is homeomorphic to a $(2n-3)$-dimensional sphere;
(ii) every point $x \in \partial\mathcal{M}_n$ corresponds to exactly one function $f \in \mathbf{S}$ which is called a *boundary function* for $\mathcal{M}_n$;
(iii) with the exception for a set of smaller dimension, at every point $x \in \partial\mathcal{M}_n$ there exists a normal vector satisfying the Lipschitz condition;
(iv) there exists a connected open set X_1 on $\partial\mathcal{M}_n$, such that the boundary $\partial\mathcal{M}_n$ is an analytic hypersurface at every point of X_1. The points of $\partial\mathcal{M}_n$ corresponding to the functions that give the extremum to a linear functional belong to the closure of X_1.

It is worth to note that all boundary functions have a similar structure. They map the unit disk U onto the complex plane $\mathbb{C}$ minus piecewise analytic Jordan arcs forming a tree with a root at infinity and having at most n tips. The uniqueness of the boundary functions implies that each point of $\partial\mathcal{M}_n$ (the set of first coefficients) defines the rest of coefficients uniquely.

2.2. Hamiltonian dynamics and integrability

Let us recall briefly the Hamiltonian and symplectic definitions and concepts that will be used in the sequel. There exists a vast amount of modern literature dedicated to different approaches to and definitions of *integrable systems* (see, e.g., [2,3,6,33]).

The classical definition of a *completely integrable system* in the sense of Liouville applies to a Hamiltonian system. If we can find independent conserved integrals which are pairwise involutory (vanishing Lie–Poisson bracket), this system is completely integrable (see, e.g., [2,3, 6]). That is each first integral allows us to reduce the order of the system not just by one, but by two. We formulate this definition in a slightly adopted form as follows.

A dynamical system in $\mathbb{C}^{2n}$ is called *Hamiltonian* if it is of the form

$$\dot{x} = \nabla_s H(x), \tag{1}$$

where ∇_s denotes the *symplectic gradient* given by

$$\nabla_s = \left(\frac{\partial}{\partial \bar{x}_{n+1}}, \ldots, \frac{\partial}{\partial \bar{x}_{2n}}, -\frac{\partial}{\partial x_1}, \ldots, -\frac{\partial}{\partial x_n}\right).$$

The function H in (1) is called the *Hamiltonian function* of the system. It is convenient to redefine the coordinates $(x_{n+1}, \ldots, x_{2n}) = (\psi_1, \ldots, \psi_n)$, and rewrite the system (1) as

$$\dot{x}_k = \frac{\partial H}{\partial \bar{\psi}_k}, \qquad \dot{\bar{\psi}}_k = -\frac{\partial H}{\partial x_k}, \quad k = 1, 2, \ldots, n. \tag{2}$$

The system has n degrees of freedom. The two-form $\omega = \sum_{k=1}^{n} dx \wedge d\bar{\psi}$ admits the Lie–Poisson bracket $[\cdot,\cdot]$

$$[f, g] = \sum_{k=1}^{n} \left(\frac{\partial f}{\partial x_k}\frac{\partial g}{\partial \bar{\psi}_k} - \frac{\partial f}{\partial \bar{\psi}_k}\frac{\partial g}{\partial x_k}\right)$$

associated with ω. The symplectic pair $(\mathbb{C}^{2n}, \omega)$ defines the Poisson manifold $(\mathbb{C}^{2n}, [\cdot,\cdot])$. These notations may be generalized for a symplectic manifold and a Hamiltonian dynamical system on it.

The system (2) may be rewritten as

$$\dot{x}_k = [x_k, H], \qquad \dot{\bar{\psi}}_k = [\bar{\psi}_k, H], \quad k = 1, 2, \ldots, n, \tag{3}$$

and the *first integrals* L of the system are characterized by

$$[L, H] = 0. \tag{4}$$

In particular, $[H, H] = 0$, and the Hamiltonian function H is an integral of the system (1). If the system (3) has n functionally independent integrals $L_1, \ldots, L_n$, which are pairwise involutory $[L_k, L_j] = 0$, $k, j = 1, \ldots, n$, then it is called *completely integrable* in the sense of Liouville. The function H is included in the set of the first integrals. The classical theorem of Liouville and Arnold [2] gives a complete description of the motion generated by the completely integrable system (3). It states that such a system admits action-angle coordinates around a connected regular compact invariant manifold.

If the Hamiltonian system admits only $1 \leqslant k < n$ independent involutory integrals, then it is called *partially integrable*. The case $k = 1$ is known as the Poincaré–Lyapunov theorem which states that a periodic orbit of an autonomous Hamiltonian system can be included in a one-parameter family of such orbits under a non-degeneracy assumption. A bridge between these two extremal cases $k = 1$ and $k = n$ has been proposed by Nekhoroshev [22] and proved later in [5,11,12]. The result states the existence of k-parameter families of tori under suitable non-degeneracy conditions.

2.3. Hamiltonian system for the coefficients

The Löwner–Kufarev parametric method (see, e.g., [10,25]) is based on a representation of any function f from the class $\mathbf{S}$ by the limit

$$f(z) = \lim_{t\to\infty} e^t w(z,t), \tag{5}$$

where the function

$$w(z,t) = e^{-t} z\left(1 + \sum_{n=1}^{\infty} c_n(t) z^n\right)$$

is a solution to the Löwner–Kufarev equation

$$\frac{dw}{dt} = -wp(w,t), \tag{6}$$

with the initial condition $w(z,0) \equiv z$. The function $p(z,t) = 1 + p_1(t)z + \cdots$ is holomorphic in U and has the positive real part for all $z \in U$ almost everywhere in $t \in [0,\infty)$. If $f \in \tilde{\mathbf{S}}$, then

$$\begin{aligned}\dot{c}_n &= c_n - \frac{e^t}{2\pi i}\int_{S^1} w(z,t)p\big(w(z,t),t\big)\frac{dz}{z^{n+2}}\\ &= -\frac{1}{2\pi i}\int_{S^1}\sum_{k=1}^{n} e^{-kt}\big(e^t w\big)^{k+1} p_k \frac{dz}{z^{n+2}}, \quad n \geqslant 1.\end{aligned} \tag{7}$$

In particular,

$$\begin{aligned}\dot{c}_1 &= -e^{-t}p_1,\\ \dot{c}_2 &= -2e^{-t}p_1c_1 - e^{-2t}p_2,\\ \dot{c}_3 &= -e^{-t}p_1\big(2c_2 + c_1^2\big) - 3e^{-2t}p_2c_1 - e^{-3t}p_3,\\ &\vdots\end{aligned}$$

We consider an adjoint vector

$$\psi(t) = \begin{pmatrix}\psi_1(t)\\ \vdots\\ \psi_n(t)\end{pmatrix},$$

with complex-valued coordinates $\psi_1,\ldots,\psi_n$, and the complex Hamiltonian function

$$H(a,\psi,u) = \sum_{k=1}^{n}\bar{\psi}_k\left(c_k - \frac{e^t}{2\pi i}\int_{S^1} w(z,t)p\big(w(z,t),t\big)\frac{dz}{z^{k+2}}\right).$$

To come to the Hamiltonian formulation for the coefficient system we require that $\bar{\psi}$ satisfies the adjoint to (7) system of differential equations

$$\dot{\bar{\psi}}_j = -\frac{\partial H}{\partial c_j}, \quad 0 \leqslant t < \infty,$$

or

$$\dot{\bar{\psi}}_j = -\bar{\psi}_j + \frac{1}{2\pi i}\sum_{k=1}^{n}\bar{\psi}_k \int\limits_{S^1}(p + wp')\frac{dz}{z^{k-j+1}}, \quad j = 1, \dots, n-1, \tag{8}$$

and

$$\dot{\bar{\psi}}_n = 0. \tag{9}$$

In particular, for $n = 3$ we have

$$\begin{aligned}
\dot{\bar{\psi}}_1 &= 2e^{-t}p_1\bar{\psi}_2 + \left(2e^{-t}p_1c_1 + 3e^{-2t}p_2\right)\bar{\psi}_3,\\
\dot{\bar{\psi}}_2 &= 2e^{-t}p_1\bar{\psi}_3,\\
\dot{\bar{\psi}}_3 &= 0.
\end{aligned}$$

2.4. First integrals and partial integrability

Let us construct the following series

$$\sum_{k=1}^{n}\bar{v}_{n-k+1}z^{k-1} = e^t w'(z,t)\sum_{k=1}^{n}\bar{\psi}_{n-k+1}z^{k-1} + e^t w'(z,t)\sum_{k=n}^{\infty}b_k z^k. \tag{10}$$

Taking into account (8) and the formula for the derivative

$$\frac{\partial(e^t w')}{\partial t} = e^t w'\big(1 - p(w,t) - wp'(w,t)\big),$$

we come to the conclusion that $\dot{\bar{v}} = 0$ and $\bar{v}$ is constant. We denote by $(L_1, \dots, L_n)^T$ the vector of the first integrals of the Hamiltonian system (7)–(9) given by

$$\begin{pmatrix} L_1\\ L_2\\ L_3\\ \dots\\ L_n \end{pmatrix} = \begin{pmatrix} 1 & 2c_1 & \dots & (n-1)c_{n-2} & nc_{n-1}\\ 0 & 1 & \dots & (n-2)c_{n-3} & (n-1)c_{n-2}\\ 0 & 0 & \dots & (n-3)c_{n-4} & (n-2)c_{n-3}\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{\psi}_1\\ \bar{\psi}_2\\ \bar{\psi}_3\\ \dots\\ \bar{\psi}_n \end{pmatrix}. \tag{11}$$

Indeed, the equality (10) implies that $L_k = \bar{v}_k$ are constants for all t and $k = 1, \dots, n$. Naturally,

$$[L_j, H] = \sum_{k=1}^{n}\frac{\partial L_j}{\partial c_k}\frac{\partial H}{\partial \bar{\psi}_k} - \frac{\partial L_j}{\partial \bar{\psi}_k}\frac{\partial H}{\partial c_k} = \sum_{k=1}^{n}\frac{\partial L_j}{\partial c_k}\dot{c}_k + \frac{\partial L_j}{\partial \bar{\psi}_k}\dot{\bar{\psi}}_k = \dot{L}_j = 0.$$

The commutator relations are:

$$[L_j, L_k] = (j-k)L_{k+j}, \quad \text{when } k+j \leqslant n, \tag{12}$$

or 0 otherwise. This implies that:

- the first integrals $(L_{[(n+1)/2]}, \dots, L_n)$ are pairwise involutory;
- the integrals $(L_1, \dots, L_{[(n-1)/2]})$ are not pairwise involutory but their Lie–Poisson brackets give all the rest of integrals.

Here $[\cdot]$ within the index field means the integer part. It is clear from the form of the matrix in the above representation of L_k, $k = 1, \dots, n$, that all these integrals are algebraically (even linearly) independent. Therefore, the Hamiltonian system (7)–(9) is partially integrable in the Liouville sense. In particular for $n = 3$, we compute

$$\begin{aligned}
\psi_1 &= \left(4c_1^2 - 3c_2\right)v_3 - 2c_1 v_2 + v_1, \\
\psi_2 &= -2c_1 v_3 + v_2, \\
\psi_3 &= v_3.
\end{aligned}$$

Remark. All previous considerations we did for the class $\tilde{\mathbf{S}}$ because it will be important for us in the future sections. But the result on partial integrability is still valid for the whole class $\mathbf{S}$ going inside the unit disk by $f \to \frac{1}{r} f(rz)$, and letting $r \to 1$.

Remark. The complete integration of this Hamiltonian system requires additional information on the trajectories, in particular, on the controls $p_1, p_2, \dots$. One way to perform such integration as a solution of the extremal problem of finding the boundary hypersurfaces of $\mathcal{M}_n$ by optimal control methods, see [27].

Remark. In view of Hamiltonian mechanics, our Hamiltonian system describes trivial motion with the constant velocity because the Hamiltonian function is linear with respect to ψ. An attempt to get a non-trivial description of the Löwner–Kufarev motion was launched in [32] by intaking a special Lagrangian. Further on in this paper, we shall give another non-trivial Hamiltonian and Lagrangian descriptions based on the sub-Riemannian geometry led on $\mathcal{M}_n$.

Remark. The coefficient bodies $\mathcal{M}_1$, $\mathcal{M}_2, \dots$ generate a hierarchy of Hamiltonian systems (7)–(8).

3. Virasoro algebra and Kirillov's operators

A *Killing vector field* is a vector field on a Riemannian manifold that preserves the metric. Killing fields are the infinitesimal generators of isometries; that is, flows generated by Killing fields are continuous isometries of the manifold. A *Witt algebra* is the Lie algebra of Killing vector fields defined on the Riemann sphere. The basis for these Killing fields is given by the holomorphic fields

$$L_n = -z^{n+1}\frac{\partial}{\partial z}.$$

The Lie–Poisson bracket of two Killing fields is

$$[L_m, L_n] = (n-m)z^{m+n+1}\frac{\partial}{\partial z} = (m-n)L_{m+n}. \tag{13}$$

The *Virasoro algebra* is the central extension of the Witt algebra by $\mathbb{C}$. The Lie–Poisson bracket for the basis vectors of the Virasoro algebra is

$$[L_m, L_n]_{Vir} = (m-n)L_{m+n} + \frac{c}{12}n\big(n^2-1\big)\delta_{n,-m}.$$

The constant $c \in \mathbb{C}$ is known as the *central charge* and is a constant of the theory.

To analyze and to represent this central extension we consider real vector fields over the unit circle. We denote the Lie group of C^∞ sense preserving diffeomorphisms of the unit circle S^1 by $\operatorname{Diff} S^1$. Each element of $\operatorname{Diff} S^1$ is represented as $z = e^{i\alpha(\theta)}$ with a monotone increasing C^∞ real-valued function $\alpha(\theta)$, such that $\alpha(\theta+2\pi) = \alpha(\theta)+2\pi$. The Lie algebra for $\operatorname{Diff} S^1$ is identified with the Lie algebra $\operatorname{Vect} S^1$ of smooth (C^∞) tangent vector fields to S^1, the infinitesimal action is $\theta \to \theta + \varepsilon\phi(\theta)$. To ϕ we associate the vector field $\phi\frac{d}{d\theta}$, and the Lie–Poisson bracket is given by

$$[\phi_1, \phi_2] = \phi_1\phi_2' - \phi_2\phi_1'.$$

Fixing the trigonometric basis in $\operatorname{Vect} S^1$, the commutator relations admit the form

$$\begin{aligned}
[\cos n\theta, \cos m\theta] &= \frac{n-m}{2}\sin(n+m)\theta + \frac{n+m}{2}\sin(n-m)\theta,\\
[\sin n\theta, \sin m\theta] &= \frac{m-n}{2}\sin(n+m)\theta + \frac{n+m}{2}\sin(n-m)\theta,\\
[\sin n\theta, \cos m\theta] &= \frac{m-n}{2}\cos(n+m)\theta - \frac{n+m}{2}\cos(n-m)\theta.
\end{aligned}$$

The space $\operatorname{Vect} S^1$ with so given Lie–Poisson bracket is the space of left-invariant vector fields.

Let I and G be Lie algebras. An exact sequence is a sequence of objects and morphisms between them, such that the image of one morphism equals the kernel of the next. Let us consider the exact sequence of Lie algebras

$$0 \to I \xrightarrow{f} E \xrightarrow{g} G \to 0.$$

E is called the *central extension* of G by I if I belongs to the center of E. The central extension is given as $E \simeq G \oplus I$. A simple example is $[x+a]_E = [x,y]_G + [a,b]_I$. The (real) Virasoro algebra is the unique (up to isomorphism) non-trivial central extension of $\operatorname{Vect} S^1$ by $\mathbb{R}$ given by the *Gelfand–Fuchs cocycle* [13]:

$$\omega(\phi_1, \phi_2) = \frac{1}{2\pi}\int_0^{2\pi}\big(\phi_1'\phi_2'' - \phi_1''\phi_2'\big)\,d\theta.$$

The Virasoro algebra *Vir* is a Lie algebra over the space $\operatorname{Vect} S^1 \oplus \mathbb{R}$ defined by the commutator

$$\big[(\phi_1, a), (\phi_2, b)\big]_{Vir} = \left([\phi_1, \phi_2]_{\operatorname{Vect} S^1}, \frac{c}{12}\omega(\phi_1, \phi_2)\right),$$

where a and b are elements of the center, $ab - ba$ vanishes, and $c \in \mathbb{R}$ is the central charge. Integration by parts leads to the 2-cocycle condition

$$\omega\big(\phi_1, [\phi_2, \phi_3]\big) + \omega\big(\phi_2, [\phi_3, \phi_1]\big) + \omega\big(\phi_3, [\phi_1, \phi_2]\big) = 0,$$

and

$$\omega(\phi_1, \phi_2) = -\frac{1}{4\pi}\int_0^{2\pi} \big(\phi_1' + \phi_1'''\big)\phi_2 \, d\theta. \tag{14}$$

Correspondingly, we consider the group $\operatorname{Diff} S^1$. The *Virasoro–Bott group* is the unique (up to isomorphism) non-trivial central extension of $\operatorname{Diff} S^1$ given by the Thurston–Bott cocycle [7]

$$\Omega(f, g) = \frac{1}{2\pi}\int_0^{2\pi} \log\big((f \circ g)'\big)\, d\log(g').$$

The Virasoro–Bott group is given by the following product on $\operatorname{Diff} S^1 \times \mathbb{R}$:

$$(f, \alpha)(g, \beta) = \left(f \circ g, \alpha + \beta + \frac{c}{12}\Omega(f, g)\right).$$

We shall identify Vect S^1 with the functions with vanishing mean value over S^1. It gives

$$\phi(\theta) = \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta.$$

Let us define a complex structure by the operator

$$J(\phi)(\theta) = \sum_{n=1}^{\infty} -a_n \sin n\theta + b_n \cos n\theta.$$

Then $J^2 = -id$. On $\operatorname{Vect} S^1 \oplus \mathbb{C}$, the operator J diagonalizes and we have

$$\phi \to \phi - iJ(\phi) = \sum_{n=1}^{\infty} (a_n - ib_n)e^{in\theta},$$

and the latter extends into the unit disk as a holomorphic function.

Taking the basis of $\operatorname{Vect} S^1 \oplus \mathbb{C}$ as $e_n = -ie^{in\theta}\partial$ we get

$$[e_n, e_m] = (n-m)e_{n+m} + \frac{c}{12}n\left(n^2-1\right)\delta_{n,-m}.$$

The Virasoro algebra is realizable both as a central extension of the Witt algebra and as an algebra of the Virasoro generators in Conformal Field Theory.

There is no general theory of infinite-dimensional Lie groups, example of which is under consideration. The interest to the particular case $\operatorname{Diff} S^1$ comes first of all from the two-dimensional Conformal Field Theory where the algebra of energy momentum tensor deformed by a central extension due to the conformal anomaly is represented by the Virasoro algebra. Entire necessary background for the construction of the theory of unitary representations of $\operatorname{Diff} S^1$ is found in the study of Kirillov's homogeneous Kählerian manifold $\operatorname{Diff} S^1/S^1$. The group $\operatorname{Diff} S^1$ acts as a group of translations on the manifold $\operatorname{Diff} S^1/S^1$ with S^1 as a stabilizer. The Kählerian geometry of $\operatorname{Diff} S^1/S^1$ has been described by Kirillov and Yuriev in [17]. The manifold $\operatorname{Diff} S^1/S^1$ admits several representations, in particular, in the space of smooth probability measures, symplectic realization in the space of quadratic differentials. We shall use its analytic representation by $\tilde{\mathbf{S}}$ and $\mathcal{M}$ mentioned in Introduction.

The Kirillov infinitesimal action of $\operatorname{Vect} S^1$ on $\tilde{\mathbf{S}}$ is given by the Goluzin–Schiffer variational formulas which lift the actions from the Lie algebra $\operatorname{Vect} S^1$ onto $\tilde{\mathbf{S}}$. Let $f \in \tilde{\mathbf{S}}$ and let $\nu(e^{i\theta})$ be a C^∞ real-valued function in $\theta \in (0, 2\pi]$ from $\operatorname{Vect} S^1$ making an infinitesimal action as $\theta \mapsto \theta + \varepsilon\nu(e^{i\theta})$. Let us consider a variation of f given by

$$\delta_\nu f(z) = \frac{f^2(z)}{2\pi i} \int\limits_{S^1} \left(\frac{wf'(w)}{f(w)}\right)^2 \frac{\nu(w)\,dw}{w(f(w)-f(z))}. \tag{15}$$

Kirillov and Yuriev [17,18] (see also [1]) have established that the variations $\delta_\nu f(\zeta)$ are closed with respect to the commutator (13) and the induced Lie algebra is the same as $\operatorname{Vect} S^1$. Moreover, Kirillov's result [15] states that there is an exponential map $\operatorname{Vect} S^1 \to \operatorname{Diff} S^1$ such that the subgroup S^1 coincides with the stabilizer of the map $f(z) \equiv z$ from $\tilde{\mathbf{S}}$.

Taking the complexification $\operatorname{Vect}_{\mathbb{C}} S^1$ of $\operatorname{Vect} S^1$ and the basis $\nu = -iz^k$ in the integrand of (15) we calculate the residue in (15) and obtain

$$L_k(f)(z) = \delta_\nu f(z) = z^{k+1}f'(z), \quad k = 1, 2, \ldots.$$

In terms of the affine coordinates in $\mathcal{M}$ we get

$$L_j = \partial_j + \sum_{k=1}^{\infty}(k+1)c_k\partial_{j+k},$$

or truncating

$$L_j = \partial_j + \sum_{k=1}^{n-j}(k+1)c_k\partial_{j+k}, \tag{16}$$

on $\mathcal{M}_n$, where $\partial_k = \partial/\partial c_k$. Considering the adjoint vector ψ (Section 2) as the vector of affine coordinates, we conclude that the vector fields given by the first integrals L_k, see (11), are exactly

Kirillov's operators. Given a fixed central charge c, Neretin [23] introduced the sequence of polynomials P_n defined by the following recurrence relations:

$$L_k(P_j) = (j+k)P_{j-k} + \frac{c}{12}k\left(k^2-1\right)\delta_{j,k}, \quad P_0 \equiv P_1 \equiv 0,\ P_j(0) = 0.$$

Representing the momentum-energy tensor in the 2D Conformal Field Theory the Schwarzian derivative naturally comes into play in the definition of P_n. It turns out that

$$\frac{cz^2}{12} S_f(z) = \sum_{n=0}^{\infty} P_n(c_1, \ldots, c_n) z^n,$$

where

$$S_f(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

is the Schwarzian derivative of a univalent function $f \in \tilde{\mathbf{S}}$. In particular,

$$\frac{1}{c}P_2(c_1, c_2) = \frac{1}{2}\left(c_2 - c_1^2\right), \qquad \frac{1}{c}P_3(c_1, c_2, c_3) = 2\left(c_3 - 2c_1c_2 + c_1^3\right),$$

$$\frac{1}{c}P_4(c_1, c_2, c_3, c_4) = 5c_4 - 10c_1c_3 - 6c_2^2 + 17c_1^2c_2 - 6c_1^4, \qquad \ldots$$

Remark. In general, we have real vector fields in $\operatorname{Vect} S^1$. The computation of L_k must be carried out with respect to the basis $1, e^{\pm ki\theta}$ that leads also to L_k with $k \leqslant 0$. However, we deal with holomorphic functions and L_k with $k > 0$ are to be treated as complex vector fields (see discussion in [16, p. 738], [1, pp. 632–634]).

4. Sub-Riemannian geometry of $\mathcal{M}_n$

A *sub-Riemannian structure* on an n-dimensional manifold $\mathcal{M}_n$ is a smoothly varying distribution $\mathcal{D}$ of k-planes together with a smoothly varying scalar product on these planes. The distribution $\mathcal{D}$ is a linear sub-bundle of a tangent bundle $T\mathcal{M}_n$ of $\mathcal{M}_n$. The *dimension* of the sub-Riemannian manifold is the pair (k, n) (see, e.g., [20,30,31]). In the case $n = k$ we come to the standard Riemannian structure. If $k < n$, then several new phenomena occur, e.g., the Hausdorff dimension of $\mathcal{M}_n$ is larger than the topological dimension n, the space of paths joining two fixed points and tangent to the distribution can have singularities. Suppose that a system of vector fields $X_1, \ldots, X_k$ form an orthonormal basis of $\mathcal{D}$ with respect to an inner product $\langle\cdot,\cdot\rangle$. The pair $(\mathcal{D}, \langle\cdot,\cdot\rangle)$ is called a *sub-Riemannian metric* on $\mathcal{M}_n$. A *horizontal path* is an absolutely continuous path $\gamma : [0,1] \to \mathcal{M}_n$ with a tangent vector $\dot\gamma$ in $\mathcal{D}$: i.e., $\dot\gamma(t) = \sum_{j=1}^k u_j(t) X_j(\gamma(t))$. The length of such a path is

$$\int_{[0,1]} \sqrt{\left\langle \dot\gamma(t), \dot\gamma(t)\right\rangle}\, dt.$$

The distance between two points is the infimum of the length of horizontal paths joining them. It is called the *Carnot–Carathéodory distance* in the literature (e.g., [21]). Sub-Riemannian structures appeared in the works of Carnot on thermodynamics and Carathéodory was inspired by his ideas. Let all vector fields $X_1, \ldots, X_k$ together with their commutators form the total tangent space $T\mathcal{M}_n$. If the number of these commutators is independent of the point of $\mathcal{M}_n$, then it is said that $X_1, \ldots, X_k$ satisfy the bracket generating condition (or Hörmander's hypoellipticity condition [14]). If the manifold $\mathcal{M}_n$ is connected (what is satisfied in our case), and the bracket generating condition holds, then any two points can be connected by a smooth horizontal path [9,28].

4.1. Sub-Riemannian structure defined by Kirillov's operators

Proposition 4.1. *Let $\mathcal{M}_n$ be the n-th coefficient body and $L_1, \ldots, L_n$ be the vector fields defined by* (11), (16). *Then the system (L_1, L_2) satisfies the bracket generating condition and the distribution is $\mathcal{D} = \operatorname{span}(L_1, L_2)$.*

Proof. The commutator relations (12) imply that the vector field L_3 is a unique vector generated by L_1 and L_2 by $[L_2, L_1] = L_3$. We denote by $\mathcal{D}_1$ the vector space generated by L_3. By $\mathcal{D}_k$ we denote the vector space given by the recurrence process $\mathcal{D}_k = [\mathcal{D}, \mathcal{D}_{k-1}] \setminus \mathcal{D}_{k-1}$. Thus, $\mathcal{D}_2 = \operatorname{span}(L_4, L_5)$, $\mathcal{D}_3 = \operatorname{span}(L_6, L_7)$, etc. For even n we have the last space $\mathcal{D}_{n/2} = \operatorname{span}(L_n)$. For odd n the last space is $\mathcal{D}_{(n-1)/2} = \operatorname{span}(L_{n-1}, L_n)$. The vector spaces

$$\mathcal{D} \oplus \mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_{[n/2]} = T\mathcal{M}_n,$$

form a grading sequence in $T\mathcal{M}_n$. The number $[n/2]$ is the degree of non-holonomy. Obviously, given L_1, L_2 we construct all other vector fields L_k, $k = 3, \ldots, n$, by recurrence of commutators and

$$T\mathcal{M}_n = \operatorname{span}(L_1, \ldots, L_n). \qquad \square$$

The scalar product $\langle\cdot,\cdot\rangle$ on $\mathcal{D}$ will be defined by the Kählerian structure of $\mathcal{M}_n$. Thus, the triple $(\mathcal{M}_n, \mathcal{D}, \langle\cdot,\cdot\rangle)$ is a sub-Riemannian manifold. By abuse of notation, let us denote it simply by $\mathcal{M}_n$.

Proposition 4.2. *The Hausdorff* (*complex*) *dimension of the sub-Riemannian manifold $\mathcal{M}_n$ is equal to*

- $(\frac{n}{2} + 1)^2 - \frac{9}{4}$ *for odd n*;
- $(\frac{n}{2} + 1)^2 - 2$ *for even n*.

Proof. Let us consider the case of odd n. The complex topological dimension is $\dim_{\mathbb{C}} \mathcal{D} = 2$, $\dim_{\mathbb{C}} \mathcal{D}_1 = 1$, $\dim_{\mathbb{C}} \mathcal{D}_k = 2$, for $k \geqslant 2$. The following formula [19,24] is used to calculate the Hausdorff dimension of $\mathcal{M}_n$:

$$\dim_{\mathbb{C}} \mathcal{D} + 2\dim_{\mathbb{C}} \mathcal{D}_1 + 3\dim_{\mathbb{C}} \mathcal{D}_2 + \cdots + \left(\frac{n-1}{2} + 1\right)\dim_{\mathbb{C}} \mathcal{D}_{\frac{n-1}{2}} = \left(\frac{n}{2} + 1\right)^2 - \frac{9}{4}.$$

For even n we observe that the dimension of the last subspace is 1. $\square$

Proposition 4.3. *A path $\gamma(s) = (c_1(s), \dots, c_n(s))$ in $\mathcal{M}_n$ is horizontal if and only if*

$$\begin{aligned} \dot{c}_3(s) &= 3c_2(s)\dot{c}_1(s) + 2c_1(s)\big(\dot{c}_2(s) - 2c_1(s)\dot{c}_1(s)\big), \\ &\ \ \vdots \\ \dot{c}_n(s) &= nc_{n-1}(s)\dot{c}_1(s) + (n-1)c_{n-2}(s)\big(\dot{c}_2(s) - 2c_1(s)\dot{c}_1(s)\big). \end{aligned} \tag{17}$$

Proof. The tangent vector to $\gamma(s)$ in the local affine basis $\partial_1, \dots, \partial_n$ is

$$\dot{\gamma}(s) = \dot{c}_1(s)\partial_1 + \dots + \dot{c}_n(s)\partial_n.$$

Let us rewrite the tangent vector $\dot{\gamma}(s)$ in the local basis L_1, L_2 of the distribution $\mathcal{D}$. We get

$$\begin{aligned} \dot{\gamma}(s) &= \dot{c}_1(s)\partial_1 + \dots + \dot{c}_n(s)\partial_n \\ &= \dot{c}_1(s)(\partial_1 + 2c_1\partial_2 + \dots + nc_{n-1}\partial_n) \\ &\quad + \big(\dot{c}_2(s) - 2c_1\dot{c}_1\big)\big(\partial_2 + 2c_1\partial_3 + \dots + (n-1)c_{n-2}\partial_n\big) \\ &\quad - \dot{c}_1(s)(2c_1\partial_2 + \dots + nc_{n-1}\partial_n) \\ &\quad - \big(\dot{c}_2(s) - 2c_1\dot{c}_1\big)\big(2c_1\partial_3 + \dots + (n-1)c_{n-2}\partial_n\big) \\ &\quad + 2c_1\dot{c}_1\partial_2 + \dot{c}_3(s)\partial_3 + \dots + \dot{c}_n(s)\partial_n \\ &= \dot{c}_1(s)L_1\big(\gamma(s)\big) + \big(\dot{c}_2(s) - 2c_1(s)\dot{c}_1(s)\big)L_2\big(\gamma(s)\big) \\ &\quad + \big(\dot{c}_3(s) - 3c_2(s)\dot{c}_1(s) - 2c_1(s)(\dot{c}_2 - 2c_1\dot{c}_1)\big)\partial_3 + \cdots \\ &\quad + \big(\dot{c}_n(s) - nc_{n-1}\dot{c}_1(s) - (n-1)c_{n-2}\big(\dot{c}_2(s) - 2c_1\dot{c}_1\big)\big)\partial_n. \end{aligned}$$

To simplify the calculations we use the notation $u_1 = \dot{c}_1$, $u_2 = \dot{c}_2(s) - 2c_1(s)\dot{c}_1(s)$, and $g_k = \dot{c}_k(s) - kc_{k-1}(s)\dot{c}_1(s) - (k-1)c_{k-2}(s)(\dot{c}_2 - 2c_1\dot{c}_1) = \dot{c}_k(s) - kc_{k-1}(s)u_1 - (k-1)c_{k-2}(s)u_2$. Then

$$\dot{\gamma}(s) = u_1L_1 + u_2L_2 + g_3L_3 + (-2g_3c_1 + g_4)\partial_4 + \dots + \big(-(n-2)g_3c_{n-3} + g_n\big)\partial_n.$$

Since the path γ is supposed to be horizontal, we get $g_3 = 0$. Continuing for the forth coordinate in the basis $L_1, \dots, L_n$, we obtain

$$\dot{\gamma}(s) = u_1L_1 + u_2L_2 + g_4L_4 + (-2g_4c_1 + g_5)\partial_5 + \dots + \big(-(n-3)g_4c_{n-4} + g_n\big)\partial_n.$$

To obtain the horizontal curve we take $g_4 = 0$. Proceeding in the same way we conclude that a horizontal path satisfies the conditions (17). □

Remark. Since we study left-invariant actions of L_k on $\mathcal{M}_n$ we can take the vanishing initial conditions $c_k(0) = 0$. So we may choose freely two coordinates c_1 and c_2 as two degrees of freedom. The resting coordinates will be given as a solution to (17).

Remark. Proposition 4.3 gives a description of horizontal paths locally in a neighborhood of the origin in $\mathcal{M}_n$. Checking the condition of horizontality (17) we must be sure that the path lies inside $\mathcal{M}_n$. The Löwner–Kufarev representation guarantees us this. For example, for $n = 3$, any Löwner–Kufarev trajectory in $\mathcal{M}_3$ corresponding to an odd function $f(z) = z + c_2 z^3 + c_4 z^5 + \cdots$ is horizontal. Just to make a concrete example, take the starlike function

$$w(z,t) = \frac{e^{-t} z}{\sqrt{1 + z^2(1 - e^{-2t})}},$$

with $p_1 \equiv 0$, $p_2 \equiv 1$, $p_3 \equiv p_4 \equiv \cdots \equiv 0$, and $c_1(t) \equiv 0$, $c_2(t) = \frac{1}{2}(e^{-2t} - 1)$, $c_3(t) \equiv 0$, etc.

4.2. Hamiltonian formalism for $\mathcal{M}_n$

We choose the symplectic scalar product for $L_1, \ldots, L_n$ to be given by the unit matrix $\{\delta_{j,k}\}$. Being restricted onto the distribution $\mathcal{D}$ and taking into account the above matrix we get the Hamiltonian in the form

$$H(\xi_1, \ldots, \xi_n, c_1, \ldots, c_n) = |l_1|^2 + |l_2|^2,$$

where

$$\begin{aligned} l_1 &= \bar{\xi}_1 + 2c_1\bar{\xi}_2 + \cdots + nc_{n-1}\bar{\xi}_n, \\ l_2 &= \bar{\xi}_2 + 2c_1\bar{\xi}_3 + \cdots + (n-1)c_{n-2}\bar{\xi}_n. \end{aligned}$$

Observe the similarity in formal variables were $\bar{\psi}_k = \partial_k = \bar{\xi}_k$ in (11), (16).

The system of Hamiltonian equations is given by

$$\begin{aligned} \dot{c}_1 &= \bar{l}_1, \\ \dot{c}_2 &= 2c_1\bar{l}_1 + \bar{l}_2, \\ \dot{c}_k &= kc_{k-1}\bar{l}_1 + (k-1)c_{k-2}\bar{l}_2, \quad k = 3, \ldots, n, \\ \dot{\xi}_k &= -(k+1)\xi_{k+1}l_1 - (k+1)\xi_{k+2}l_2, \quad k = 1, \ldots, n-2, \\ \dot{\xi}_{n-1} &= -n\xi_n l_1, \\ \dot{\xi}_n &= 0. \end{aligned} \tag{18}$$

Proposition 4.4. *Any solution of the Hamiltonian system* (18) *is a horizontal path.*

Proof. Observe that

$$\bar{l}_1 = \dot{c}_1 \quad \text{and} \quad \bar{l}_2 = \dot{c}_2 - 2c_1\dot{c}_1. \tag{19}$$

Substituting $\bar{l}_1$ and $\bar{l}_2$ into equations for $\dot{c}_3, \ldots, \dot{c}_n$, we obtain the horizontality conditions (17). □

Likely for horizontal paths, we assume vanishing initial conditions.

Proposition 4.5. *Define* l_3 *as*

$$l_3 = \bar{\xi}_3 + 2c_1\bar{\xi}_4 + \cdots + (n-2)c_{n-3}\bar{\xi}_n.$$

Then,

(i) $\dot{l}_1 = \bar{l}_2 l_3$ *and* $\dot{l}_2 = -\bar{l}_1 l_3$.
(ii) *The energy of the system* $\frac{1}{2}(|u_1|^2 + |u_2|^2)$ *is conserved along the geodesics. The Carnot–Carathéodory length of the tangent vector is conserved along the geodesics.*

Proof. The proof of (i) is straightforward. Differentiating l_1 and l_2 and using expressions for $\dot{\xi}_k$ and $\dot{c}_k$ from the Hamiltonian system (18) we obtain the necessary result. To prove (ii) we observe that $\frac{\partial}{\partial t}(|l_1|^2 + |l_2|^2) = 0$ by (i). Moreover, the values of u_1 and u_2 coincide with $\bar{l}_1$ and $\bar{l}_2$ on geodesics by (19). □

As a consequence we get the solution to (18) for $n = 3$. Observe that $l_3 = \bar{\xi}_3 = const$ in this case. Hence, $\ddot{c}_1 = \dot{\bar{l}}_1 = l_2\xi_3 = \overline{(\dot{c}_2 - 2c_1\dot{c}_1)}\xi_3$ by the above proposition. We continue by $\ddot{c}_2 = \frac{d^2}{dt^2}(c_1^2) + \dot{\bar{l}}_2 = \frac{d^2}{dt^2}(c_1^2) - l_1\xi_3 = \frac{d^2}{dt^2}(c_1^2) - \dot{\bar{c}}_1\xi_3$. Therefore,

$$\begin{aligned} \ddot{c}_1 + |\xi_3|^2 c_1 &= \bar{K}\xi_3, \\ \dot{c}_2 &= 2c_1\dot{c}_1 - \bar{c}_1\xi_3 + K, \end{aligned} \tag{20}$$

where K is a constant of integration and is calculated by the initial speed $K = \dot{c}_2(0)$. The solution to (20) is

$$c_1 = Ae^{i|\xi_3|t} + Be^{-i|\xi_3|t} + \bar{K}/\bar{\xi}_3, \quad \text{where } A + B + \bar{K}/\bar{\xi}_3 = 0.$$

Substituting c_1 in the equation for c_2 we get

$$\begin{aligned} c_2 = {} & A^2 e^{2i|\xi_3|t} + B^2 e^{-2i|\xi_3|t} - 2\big(Ae^{i|\xi_3|t} + Be^{-i|\xi_3|t}\big)(A+B) \\ & - \frac{i\xi_3}{|\xi_3|}\big(Ae^{-i|\xi_3|t} - Be^{i|\xi_3|t} - (A-B)\big) + 4AB + A^2 + B^2. \end{aligned}$$

The coordinate c_3 is calculated as a solution to the equation

$$\dot{c}_3 = 3c_2\dot{c}_1 + 2c_1(\dot{c}_2 - 2c_1\dot{c}_1), \quad c_3(0) = 0.$$

The corresponding explicit expression is a matter of elementary calculations and we omit awkward formulas.

Remark. Our Hamiltonian formalism and geodesics are linked to the sub-Riemannian geometry led on $\mathcal{M}_n$ by Kirillov's vector fields. So there is no direct connection with the first Hamiltonian system described in Section 2.3. The above Hamiltonian system (18) gives local geodesics in $\mathcal{M}_n$ about the origin and we do not expect any global description of geodesics because starting from the origin they may leave $\mathcal{M}_n$ in time.

4.3. *Lagrangian formalism for $\mathcal{M}_3$*

Let us consider the Lagrangian function

$$L(c, \bar{c}, \dot{c}, \bar{\dot{c}}) = |\dot{c}_1|^2 + |\dot{c}_2 - 2c_1\dot{c}_1|^2 + \operatorname{Re} \bar{\lambda}\big(\dot{c}_3 - 3c_2\dot{c}_1 - 2c_1\dot{c}_2 + 4c_1^2\dot{c}_1\big). \tag{21}$$

It splits in two terms: the kinetic energy $|\dot{c}_1|^2 + |\dot{c}_2 - 2c_1\dot{c}_1|^2$, and the non-holonomic constraint $\dot{c}_3 = 3c_2\dot{c}_1 + 2c_1\dot{c}_2 - 4c_1^2\dot{c}_1$, that reflects the horizontality condition. We are interested in minimizing the action integral

$$S(c, \tau) = \int_0^\tau L(c, \bar{c}, \dot{c}, \bar{\dot{c}})\, ds.$$

The minimum of the action is attained at a critical curve $\zeta(s)$ satisfying the Euler–Lagrange system

$$\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{c}}\right) = \frac{\partial L}{\partial c}, \qquad \frac{d}{ds}\left(\frac{\partial L}{\partial \bar{\dot{c}}}\right) = \frac{\partial L}{\partial \bar{c}}. \tag{22}$$

Proposition 4.6. *The solution to the Euler–Lagrange system* (22) *is a solution to the Hamiltonian system* (18) *if and only if it is a horizontal path.*

Proof. If the solution to the Euler–Lagrange system (22) is a solution to the Hamiltonian system (18), then it is a horizontal path by Proposition 4.4.

To show the reciprocal statement we perform auxiliary calculation. Substituting the Lagrangian (21) in equations (22) we get

$$\begin{gathered}
\bar{\ddot{c}}_1 - 2c_1\big(\overline{\ddot{c}_2 - (\ddot{c}_1^2)}\big) - \bar{\lambda}\dot{c}_2 = 0, \\
\overline{\ddot{c}_2 - (\ddot{c}_1^2)} - \bar{\lambda}\dot{c}_1 = 0, \\
\frac{d}{ds}\bar{\lambda} = 0.
\end{gathered} \tag{23}$$

We conclude that λ is a constant. Simplifying the first two equations we get

$$\begin{gathered}
\ddot{c}_1 = \xi_3\big(\overline{\dot{c}_2 - (\dot{c}_1^2)}\big), \\
\dot{c}_2 = (\dot{c}_1^2) - \bar{\dot{c}}_1\xi_3 + K, \\
\lambda = \xi_3.
\end{gathered} \tag{24}$$

The latter equality is due to the Legendre transform. In the latter system we recognize the equations for geodesics (20). □

4.4. Dual basis

The following 1-forms give the dual basis of the cotangent space for the basis $\{L_k\}$ of the tangent space:

$$\begin{aligned}
\omega_1 &= dc_1, \\
\omega_2 &= dc_2 - 2c_1\omega_1, \\
\omega_k &= dc_k - 2c_1\omega_{k-1} - 3c_2\omega_{k-2} - \cdots - kc_{k-1}\omega_1, \quad k = 3, \ldots, \infty.
\end{aligned} \tag{25}$$

We have $\omega_k(L_j) = \delta_{kj}$. Define the forms η_k by

$$\eta_k = dc_k - kc_{k-1}\omega_1 - (k-1)c_{k-2}\omega_2, \quad k = 3, \ldots, \infty. \tag{26}$$

Then the form $\eta = \sum_{k=1}^{n}$ defines the distribution $\mathcal{D}$ for $\mathcal{M}_n$ as a kernel

$$\mathcal{D} = \{X \in T\mathcal{M}_n\colon\ \eta(X) = 0\}.$$

A *contact form* α on a $(2n+1)$-dimensional manifold is a local 1-form with the property

$$\alpha \wedge d\alpha \neq 0.$$

In our case n is the complex dimension. Nevertheless, for $n = 3$ we get

$$\eta_3 \wedge d\eta_3 = dc_1 \wedge dc_2 \wedge dc_3 \neq 0.$$

The form η_3 is contact and its kernel defines the distribution $\mathcal{D}$ in $\mathcal{M}_3$.

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